Weak Topology

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Part - 1

Requirements

1. $\mathcal{F}$-topology of X
2. Hausdorff locally topology
3. Topological vector space / locally convex space
4. Abelian topological group
5. Hausdorff vector topology
6. Natural map

Overview

We earlier discussed the following :

Given a non-empty set X and

a family $\mathcal{F}$ of functions such that each f in $\mathcal{F}$ maps X into a topological space (Y_f, τ_f),

- how to find a smallest topolgy (denoted by $\tau_{\mathcal{F}}$) for X with respect to which each member of $\mathcal{F}$ is continuous.
- results on the topology $\tau_{\mathcal{F}}$ or the $\mathcal{F}$-**topology of** X or **the topology of** X **induced by** $\mathcal{F}$**.**

Overview

We now discuss the same by considering

- X as a normed space, and
- the topologizing family $\mathcal{F}$ as the set X^* (the dual space of X with respect to the **norm topology** of X).

Already the normed space X has a topology, the **norm topology**.

The topology of X induced by X^* is a subtopology of the norm topology. It is called the **weak topology** of X.

Something to be added for the following statement: Note that weak topology can give more compact sets and convergent sequences.

Conventions

- If X is a normed space, whenever reference is made so some topological property in X without specifying the topology, **the norm topology** is implied.
- The notation X^* and the term "the dual space of X" always refer to **the dual space of X with respect to the norm topology of X**, expect where explicitly stated otherwise, even in contexts in which another topology for X is being discussed.

Weak topology of a normed space X

Definition 1.

Let X be a normed space. Then the topology for X induced by the topologizing family X^* is the **weak topology of X or the X^*-topology of X or the topology $\sigma(X, X^*)$.**

That is, the weak topology of a normed space is the smallest topology for the space such that every member of the dual space is continuous with respect to that topology.

Consequence of Hahn-Banach Theorem

Theorem 2.

If x and y are different elements of a normed space X, then there is a bounded linear functional f on X such that $f(x) \neq f(y)$.

By the preceding result, X^* (the topologizing family of functions) for X is **separating** (or **total**).

Theorem 3 (Recall).

Let X be a set and let $\mathcal{F}$ be a family of functions such that each f in $\mathcal{F}$ maps X into a topological space (Y_f, τ_f). If each Y_f is T_0, or T_1, or T_2, or T_3, or $T_{3\frac{1}{2}}$, then the $\mathcal{F}$-topology of X satisfies that same separation axiom.

Note that each Y_f is $\mathbb{F}$ with the usual topology.

Every metric space is completely regular (of course, it is normal), in particular $\mathbb{F}$.

The weak topology of a normed space X is Hausdorff. Moreover, it is a completely regular ($3\frac{1}{2}$) topology.

Theorem 4 (Recall).

Suppose that X is a vector space and that X' is a subspace of the vector space $X^{\#}$ of all linear functionals on X. Then the X'-topology of X is a locally convex topology, and the dual space of X with respect to this topology is X'.

That is,

$$(X, \tau_{X'})^* = X'. \tag{1}$$

Notation : In (1), the dual of X with respect to the topology τ is denoted by $(X, \tau)^*$.

The weak topology of a normed space X is a locally convex topology, and the dual space of X with respect to the weak topology is X^*. That is, $(X, \text{weak topology})^* = X^* = (X, \text{norm topology})^*$.

Dual of X with norm topology / weak topology

Theorem 5.

A linear functional on a normed space is continuous with respect to the weak topology if and only if it is continuous with respect to the norm topology.

That is,

$$(X, \textit{norm topology})^* = (X, \textit{weak topology})^*.$$

Continuity and weak continuity are equivalent for linear functionals on a normed space.

The convergence in $\mathcal{F}$-topology

Theorem 6.

Let X be a set and $\mathcal{F}$ a topologizing family of functions for X. Suppose that (x_α) is a net in X and x is a member of X. Then

$$x_\alpha \to x$$

with respect to the $\mathcal{F}$-topology iff

$$f(x_\alpha) \to f(x)$$

for each f in $\mathcal{F}$.

Convergence in weak topology

Theorem 7.

Let X be a normed space. Suppose that (x_α) is a net in X and x is a member of X. Then

$$x_\alpha \to x$$

with respect to the weak topology iff

$$f(x_\alpha) \to f(x)$$

for each f in X^*.

Recall

Theorem 8 (Characterization of Cauchy net with respect to X'-topology of X).

Suppose that X is a vector space and that X' is a subspace of $X^{\#}$. Let (x_α) be a net in X. Then the following are equivalent :

1. *The net (x_α) is Cauchy with respect to the X'-topology of X.*
2. *For each f in X', the net $(f(x_\alpha))$ is Cauchy in $\mathbb{F}$.*
3. *For each f in X', the net $(f(x_\alpha))$ is convergent in $\mathbb{F}$.*

Here $\mathbb{F}$ denotes the field of real or complex numbers.

Results derived for a Hausdorff locally convex topology

All of the results derived for a **Hausdorff locally convex topology** induced by a separating vector space of linear functionals hold for the weak topology of a normed space X.

In particular, if (x_α) is a net in X and x is an element of X, then

- $x_\alpha \xrightarrow{w} x$ if and only if $x^* x_\alpha \to x^* x$, for each x^* in X^*.
- (x_α) is weakly Cauchy if and only if $(x^* x_\alpha)$ is Cauchy (that is, convergent) net in $\mathbb{F}$, for each x^* in X^*.

Notations

the weak convergence of a net (x_α) to an element x	$x_\alpha \xrightarrow{w} x$ (or) $w\text{-}\lim_\alpha x_\alpha = x$
the weak convergence of a set A	$\overline{A}^w$

A topological property that holds with respect to the weak topology is said to be **weak** property or to hold **weakly**.

Weak topology may be a proper subtopology for the norm topology

The weak topology is weaker than the norm topology.

It is clear that if a sequence (x_n) converges to x in norm, then $(f(x_n))$ converges to $f(x)$, for each $f \in X^*$.

We shall see an example that $(f(x_n))$ converges to $f(x)$ for all $f \in X^*$ but (x_n) does not converge to x in norm.

Hence the weak topology is really weaker than the norm topology.

Weak topology may be a proper subtopology for the norm topology

Example 9.

Let (e_n) *be the sequence of unit vectors in* ℓ_2.
Since $x^* e_n \to 0$ *for each* x^* *in* ℓ_2^*, *the sequence* (e_n) *converges to* 0 *with respect to the weak topology. That is,* $e_n \xrightarrow{w} 0$.
Since $\|e_n\|_2 = 1$ *for each* n, *the sequence* (e_n) *cannot converge to* 0 *with respect to the norm topology. That is,* $e_n \nrightarrow 0$.

Weak topology may be a proper subtopology for the norm topology

We observed that the sequence (e_n) is not converging to 0 in norm. But if we hit the sequence (e_n) with any functional, it converges to zero. Thus it is possible for the weak topology of a normed space to be a proper subtopology for the norm topology.

Exercise 10.

The sequence (e_n) in ℓ_p $(1 \leq p < \infty)$ weakly converges to 0 but cannot converge to 0 in norm. [Hint : Example (9) and $\ell_p^* = \ell_q$, for $1 \leq p < \infty$.]

Recall

Another subbasis and basis of the topology (induced by a subspace of $X^\#$)

We have discussed the following :

- The collection
$$\left\{ f^{-1}(U) : f \in \mathcal{F}, U \in \tau_f \right\}$$
is the **standard subbasis** for the $\mathcal{F}$-topology.
- The **standard basis** for the $\mathcal{F}$-topology is the collection of all sets that are intersection of finitely many members of this subbasis.

We shall now discuss a collection of $\mathcal{F}$-open subsets of X which contains a particular point $x \in X$.

Recall

Another subbasis and basis of the topology (induced by a subspace of $X^{\#}$)

Theorem 11.

Suppose that X is a vector space and that X' is a subspace of $X^{\#}$. For each x in X and each f in X', let

$$B(x, \{f\}) = \{ y : y \in X, |f(y - x)| < 1 \}.$$

Similarly, for each x in X and each finite subset A of X', let

$$B(x, A) = \{ y : y \in X, |f(y - x)| < 1 \text{ for each } f \text{ in } A \}.$$

Recall

Another subbasis and basis of the topology (induced by a subspace of $X^{\#}$)

Theorem 12 (contd.).

Let

$$\mathfrak{S} = \{ B(x, \{f\}) : x \in X, f \in X' \}$$

and let

$$\mathfrak{B} = \{ B(x, A) : x \in X, A \text{ is a finite subset of } X' \}.$$

Then $\mathfrak{S}$ is a subbasis and $\mathfrak{B}$ a basis for the X' topology of X. If U is a subset of X that is open with respect to the X' topology and x_0 is an element of U, then there is a finite subset A_0 of X' such that $B(x_0, A_0) \subseteq U$; that is, the set U includes a basic neighborhood of x_0 that is "centered" at x_0.

Basis for the weak topology

A particularly useful basis for the weak topology of X is given by the collection of all sets of the form

$$B(x, A) = \left\{ y \in X : |x^*(y - x)| < 1 \quad \text{for each} \quad x^* \in A \right\}$$

such that $x \in X$ and A is a finite subset of X^*.

We shall discuss later that local base at 0,

$$B(0, x_1^*, x_2^*, \ldots, x_n^*) = \left\{ y \in X : |x_i^* y| < 1 \quad \text{for all} \quad 1 \leq i \leq n \right\}$$

is enough to analyse topological properties of the weak topology of X.

Shifting local base at 0 to any point is possible.

Weakly boundedness

A set in a topological vector space is called **bounded** or **von Neumann bounded**, if every neighborhood of the zero vector can be inflated to include the set.

Conversely a set that is not bounded is called **unbounded**.

The concept was first introduced by John von Neumann and Andrey Kolmogorov in 1935.

The word “bounded” makes no sense in a general topological space.

Weakly boundedness

Definition 13.

A subset A of a topological vector space is **bounded** *if, for each neighbourhood U of 0, there is a positive s_U such that*

$$A \subseteq tU$$

whenever $t > s_U$.

Definition 14.

A subset A of a normed space X is **weakly bounded** *if, for each weak neighbourhood U of 0 in X, there is a positive s_U such that*

$$A \subseteq tU$$

whenever $t > s_U$.

Recall : A result concerning boundedness with respect to $\mathcal{F}$-topology

The following is a result concerning boundedness with respect to the topology induced on a vector space X by a subspace of $X^{\#}$.

Theorem 15 (A useful test for boundedness with respect to $\mathcal{F}$-topology).

Suppose that X is a vector space and that X' is a subspace of $X^{\#}$. Then a subset A of X is bounded with respect to the X'-topololgy if and only if $f(A)$ is bounded in $\mathbb{F}$ for each f in X'.

Corollary 16 (Characterization of weakly bounded).

Let X be a normed space. Then a subset A of X is weakly bounded if and only if $x^*(A)$ is bounded in $\mathbb{F}$ for each x^* in X^*.

A is weakly bounded is equivalent to requiring that "$x^*(A)$ is a bounded set of scalars for each x^* in X^*".

A **is unbounded** iff there exists an $x^* \in X^*$ such that $x^*(A)$ is unbounded in $\mathbb{F}$.

Let us first discuss relation between (norm) bounded and weakly bounded.

Exercise 17.

Verify the following statement :
Since every weakly open set is open, every bounded set of a normed space is weakly bounded.

We have seen an example that the weak topology of a normed space can be a proper subtopology of the norm topology, so it might happen that a subset of a normed space to be weakly bounded than to be bounded.

Perhaps suprisingly, this is not the case.

Theorem 18.

A subset of a normed space is bounded iff it is weakly bounded. WT-1(P-3)

We observed that A is weakly bounded is equivalent to requiring that "$x^*(A)$ is a bounded set of scalars for each x^* in X^*". The same is equivalent to requiring "A is bounded."

Recall : Vector Topology

Definition 19.

Suppose that X is a vector space with a topology τ such that addition of vectors is a continuous operation from $X \times X$ into X and multiplicaiton of vectors by scalars is a continuous operation from $\mathbb{F} \times X$ into X.

Then τ is a **vector** *or* **linear topology** *for X, and the ordered pair (X, τ) is a* **topological vector space (TVS)** *or a* **linear topological space (LTS)**.

If τ has a basis consisting of convex sets, then τ is a **locally convex topology** *and the TVS (X, τ) is a* **locally convex space (LCS)**.

Recall : Vector Topology

The continuity of the vector space operations in a TVS creates a link between the vector space structure and the topology of the space.

The additional property possessed by an LCS provides each of its points with **a supply of nicely shaped neighbourhoods.**

Exercises 20.

1. *Prove that every norm topology is a locally convex topology.*
2. *Prove that weak topology of a normed space is a locally convex topology.*

Weak topology is a linear topology which is Hausdorff.

If $x_\alpha \xrightarrow{w} x$, then $x_\alpha + y \xrightarrow{w} x + y$ for any $y \in X$.

Shifting local base of any point to zero (or vice-versa) is possible.

Recall : Vector Topology

Theorem 21.

Every T_0 vector topology is completely regular.

Thus, a vector topology that satisfies any of the separation axioms T_0 through $T_{3\frac{1}{2}}$ actually satisfies all of them.

It is traditional that such vector topologies be called **Hausdorff**, but it should be kept in mind that for vector topologies the Hausdorff axiom is implied by the T_0 axiom and implies completely regularity.

Natural Map

Every element x of X induces a linear funtional f_x on X^* defined

$$F_x(f) = f(x)$$

for all $f \in X^*$. It is clear that F_x is a linear functional on X^*, and the map $x \mapsto F_x$ is linear.

It follows from the definition that $\|Fx\| = \|x\|$.

The Hahn-Banach theorem implies the stronger assertion that $\|F_x\| = \|x\|$.

Recall : Vector Topology

Theorem 22.

Every compact subset of a topological vector space is bounded. Thus, every convergent **sequence** *in a topological vector space is bounded.*

Theorem 23.

Every Cauchy **sequence** *in a topological vector space is bounded.*

Theorem 24.

Every convergent net in an abelian topological group is Cauchy.

Boundedness of weakly (compact sets, Cauchy sequences, convergent sequences)

Theorem 25.

Every weakly compact set in a normed space is weakly bounded.

Theorem 26.

Every weakly Cauchy **sequence** *in a normed space is weakly bounded.*

Theorem 27.

Every weakly convergent **sequence** *in a normed space is weakly Cauchy, hence weakly bounded.*

Important consequences for the continuity of linear operators

We now discuss boundedness of a linear operator between normed spaces.

Theorem 28.

Let T be a linear operator from a normed space X into a normed space Y. The following are equivalent :

1. *T is bounded.*
2. *$T(B_X)$ is bounded.*
3. *$y^*T(B_X)$ is bounded for each y^* in Y^*.*
4. *$y^*T \in X^*$ whenever y^* in Y^*.*
5. *y^*T is weakly continuous on X whenever y^* in Y^*.*

Norm-to-norm continuous and weak-to-weak continuous

We have seen that a linear functional on a normed space is continuous if and only if it is weakly continuous. That is, continuity and weak continuity are equivalent for linear functionals on a normed space.

Theorem 29.

A linear operator T from a normed space X into a normed space Y is bounded if and only if $y^*T \in X^*$ whenever $y^* \in Y^*$.

The above result tells that the linear operator T is continuous if and only if y^*T is a weakly continuous (or, continuous, both are same) linear functional on X whenever $y^* \in Y^*$.

Norm-to-norm continuous and weak-to-weak continuous

Let X, Y be normed spaces and let $T : X \to Y$ be linear. We call T **norm-to-norm continuous** if X and Y are endowed with the norm topologies and similarly, **weak-to-weak continuous** if X, Y are endowed with the weak topologies.

Recall

Theorem 30.

Let W be a topological space and let X be a set topologized by a family $\mathcal{F}$ of functions and let g be a function from W into X. Then g is continuous iff $f \circ g$ is continuous for each f in $\mathcal{F}$.

Relation between norm-to-norm continuous and weak-to-weak-continuous

Consider $W = X$ with the norm topology in the above result.

"$y^* T$ is continuous for each $y^* \in Y^*$" is same as "$y^* T$ is weakly continuous" for each $y^* \in Y^*$.

Theorem 31.

A linear operator T from a normed space X into a normed space Y is norm-to-norm continuous if and only if it is weak-to-weak continuous.
WT-2(P-3)

Weak-to-weak homeomorphism

Theorem 32.

A linear operator T from a normed space X onto a normed space Y is an (linear) isomorphism of normed spaces if and only if it is a weak-to-weak homeomorphism.

If two normed spaces are implicitly treated as the same because of some natural isometric isomorphism from one onto the other (the way c_0^* is usually identified with ℓ_1), then the weak topologies of the two spaces are preserved by the same isometric isomorphism.

We now discuss a characterization for normed spaces having identical weak and norm topologies.

Recall : A result in $\mathcal{F}$-topology concerning unboundedness of a nonempty set which is open with respect to the $\mathcal{F}$-topology

There is a dramatic difference between $\mathcal{F}$-topologies and the norm topologies when the topolgizing subspace of $X^{\#}$ is infinite-dimensional.

Theorem 33 (Recall).

Suppose that X is a vector space and that X' is a subspace of $X^{\#}$. If X' is infinite-dimensional, then every nonempty subset of X that is open with respect to the X'-topology is unbounded with respect to that topology.

Exercise 34.

Prove that if a normed space X is infinite-dimensional, then X^* is infinite-dimensional.

Theorem 35.

Every nonempty weakly open subset of an infinite-dimensional normed space is unbounded.

Every weakly open subset of X is too big. If X is infinite-dimensional, its open unit ball cannot be weakly open, **so the norm and weak topologies of X must differ.**

Hence the weak topology of an infinite-dimensional normed space is **always** a proper subtopology of the norm topology.

Characterization of normed spaces whose norm and weak topologies are identical.

Let X be an infinite-dimensional normed space. Consider the open unit ball of X.

We know that every nonempty weakly open subset of an infinite-dimensional normed space is unbounded. Hence the open unit ball cannot be weakly open (since it is bounded), so the norm and weak topologies of X must differ.

This shows that if norm and weak topologies of a normed are the same, then the space of finite dimension.

Characterization of normed spaces whose norm and weak topologies are identical.

Recall :

Theorem 36.

Suppose that X is a finite dimensional vector space. Then X has exactly one Hausdorff vector topology. This topology is induced by a Banach norm.

Theorem 37.

The norm and weak topologies of a normed space are the same if and only if the space is finite dimensional.

We proved that if norm and weak topologies of a normed are the same, then the space of finite dimension. Converse part can be proved by using Theorem (36).

Is the weak topology induced by a metric ?

We proved that the norm and weak topologies of a normed X are the same iff $\dim X < \infty$.

Therefore the weak topology of an infinite-dimensional normed space is not induced by the norm of the space.

Is the weak topology induced by any metric?

Definition 38.

A topology (a collection τ of open sets) on a given space X is called **metrizable** *if there exists a metric on X such that the open sets generated by this metric are exactly those that are members of τ.*

Is the weak topology induced by a metric ?

Sequence convergence is better than net convergence. Working with natural numbers is better than directed sets.

If a topology is metrizable, we can get rid of nets.

Can the weak topology of an infinite dimensional normed space, be metrizable?

If a topological space is not metrizable, sequences might not be adequate to detect the accumulation points.

Is the weak topology induced by a metric ?

In fact, the weak topology of such a space is not induced by any metric at all.

Theorem 39.

The weak topology of a normed space is induced by a metric if and only if the space is finite dimensional. *WT-3(P-5)*

We cannot make the weak topology of an infinite dimensional normed space metrizable. So, we cannot get rid of the nets, in general.

It is possible that some useful subsets of X could be metrizable (we shall see later).

Weakly Complete

Completeness is yet another property that the weak topology cannot have unless the normed space is finite dimensional.

Definition 40.

A net $(x_\alpha)_{\alpha \in \mathbb{I}}$ in a topological vector space X is **Cauchy** *if for every neighbourhood U of 0 in X, there exists an α_U in $\mathbb{I}$ such that*

$$x_\beta - x_\gamma \in U$$

whenver $\alpha_U \leq \beta,\ \alpha_U \leq \gamma$, and X is **complete** *if every Cauchy net in X converges.*

Proposition 41.

The weak topology of a normed space is complete if and only if the space is finite dimensional. WT-4(P-5)

We use Helly's theorem to prove the preceding result.

Relation between weakly closed and (norm) closed

We have seen that an infinite dimensional normed space X must have open convex subset (for example, the open balls in X) that are not weakly open.

Because of this, it might not seem likely that the closed convex subsets of an arbitrary normed space would have to be the same as it weakly closed convex subsets.

But surprisingly, the (norm) closure and weak closure of a convex subset of a normed space are the same, by the following result.

Theorem 42 (Mazur, 1933).

The closure and weak closure of a convex subset of a normed space are the same. In particular, a convex subset of a normed space is closed if and only if it is weakly closed.

Relation between weakly closed and (norm) closed

Corollary 43.

The closure and weak closure of a subspace of a normed space are the same, so a subspace of a normed space is closed if and only if it is weakly closed.

Exercise 44.

If A is a subset of a normed space, then

$$\overline{co}(A) = \overline{co(A)} = \overline{co(A)}^{w} = \overline{co}^{w}(A).$$

Corollary 45.

If $(x_\alpha)_{\alpha \in \mathbb{I}}$ is a net in a normed space that converges weakly to some x, then some sequence of convex combinations of members of $\left\{x_\alpha : \alpha \in \mathbb{I}\right\}$ converges to x with respect to the norm topology.

Topology of a subspace inherited from the weak topology

Suppose that M is a subspace of a normed space X. Then a statement such as "the net (x_α) in M converges weakly to an x in M" might seem ambiguous, since it is not clear whether the statement refers to the weak topology of X or to that of M treated as a normed space in its own right.

Fortunately, it makes no difference.

Proposition 46.

Let M be a subspace of a normed space X. Then the weak topology of the normed space M is the same as the topology of M inherited from the weak topology of X.

Norm function does not have to be weakly continuous.

One of the basis properties of normed spaces is that the norm function

$$x \mapsto \|x\|$$

is continuous. However, it does not have to be weakly continuous.

For instance, the sequence $(e_n)_{n=1}^{\infty}$ in ℓ_2 weakly converges to 0 but $\|e_n\| \nrightarrow 0$. Hence the norm function on ℓ_2 is not weakly continuous.

Thus, it is not always true that $\|x_\alpha\| \to \|x\|$, when a net (x_α) in a normed space converges weakly to some x.

Norm function does not have to be weakly continuous.

Exercises 47.

1. *The norm function is weakly continuous if and only if the norm and weak topologies of the space are the same, that is, if and only if, the space is finite dimensional.*
2. *Suppose that X is an infinite dimensional normed space. Show that there is a net in S_X that converges weakly to 0.*

   Notice that this implies that the map $x \mapsto \|x\|$ from a normed space into $\mathbb{F}$ is not weakly continuous if the normed space is infinite dimensional.

Norm function is wekly lower semicontinuous.

Theorem 48.

If (x_α) *is a weakly convergent net in a normed space then*

$$\|w\text{-}\lim_\alpha x_\alpha\| \leq \lim inf_\alpha \|x_\alpha\| = \lim_\alpha \inf\left\{\|x_\beta\| : \alpha \leq \beta\right\}.$$

Here "$w\text{-}\lim_\alpha$*" is the weak limit of* (x_α)*.*

Definition 49.

A function f *from a topological space* X *into* $\mathbb{R}$ *is said to be* **lower semi-continuous** *if*

$$f(x) \leq \liminf_\alpha f(x_\alpha)$$

whenever (x_α) *is a net in* X *converging to some element* x *of* X*.*

Thus, the preceding theorem says that norm functions are weakly lower semi-continuous.

Sequential properties of the weak topology

We proved that the weak topology of a normed space is complete if and only if the space is finite dimensional.

Hence every infinite-dimensional normed space contains a weakly Cauchy net with no weak limit, but that does not eliminate the possibility that all of the weakly Cauchy **sequences** in such a space could be weakly convergent. This sometimes happens.

Definition 50.

A normed space is **weakly sequentially complete** *if every weakly Cauchy* **sequence** *in the space has a weak limit.*

Convergent Sequences

We have discussed that convergence of a sequence implies weak convergence, but the converse **need not be true** if the space is infinite dimensional.

One may ask whether weak convergence and convergence are different in every infinite dimensional space. The answer is in the negative.

Theorem 51 (Schur's lemma).

Every weakly **Cauchy sequence** *in* ℓ_1 *is norm convergent.*

Since every weakly convergent sequence is weakly Cauchy, the space ℓ_1 is weakly sequentially complete. But is cannot be weakly complete.

Schur's property

Since weakly convergent sequences are weakly Cauchy, every weakly convergent sequences in ℓ_1 is actually norm convergent to the weak limit of the sequence.

The fact that ℓ_1 possesses the property first appeared in a 1920 paper by J. Schur.

Definition 52.

A normed space has **Schur's property** *if it satisfies the following condition : Whenever (x_n) is a* **sequence** *in the space and x an element of the space such that*

$$x_n \xrightarrow{w} x, \quad \textit{it follows that} \quad x_n \to x.$$

The space ℓ_2 does not have Schur's property but it does have the following weakened version of the property.

Definition 53.

A normed space has the **Radon-Riesz property or the Kadets-Klee property or property (H)**, *and is called a* **Radon-Riesz space**, *if it satisfies the following condition:*

Whenever (x_n) is a sequence in the space and x an element of the space such that $x_n \xrightarrow{w} x$ and $\|x_n\| \to \|x\|$, it follows that $x_n \to x$.

Example 54.

The space ℓ_2 has the Radon-Riesz property.

Exercises 55.

1. *Prove that an infinite-dimensional normed space with its weak topology is of the first category in itself.*
2. *Prove that a normed space with its weak topology is of the second category in itself if and only if the space is finite-dimensional.*
3. *Show that c_0 is not weakly sequentially complete.*
4. *Suppose that $1 < p < \infty$. Show that ℓ_p is weakly sequentially complete.*
5. *Show that the spaces c_0 and ℓ_p such that $1 < p < \infty$ all lack Schur's property.*

Summary

If X is an infinite dimensional normed space, then

1. every nonempty weakly open subsets of X is unbounded.
2. weak and norm topologies are different (converse is also true).
3. weak topology is not metrizable (converse is also true).
4. weak topology is not complete (converse is also true).
5. it may be weakly sequentially complete (for example, the space ℓ_1).
6. norm function is not weakly continuous (converse is also true).
7. weak topology is of the first category in itself.
8. weak topology is of the second category in itself (converse is also true).

References

1. Robert E. Megginson, An Introduction to Banach Space Theory, Springer, 1991.
2. Stijn Vermeeren, Sequences and nets in topology, University of Leeds, Lecture Notes, 2010.